

CANONICAL FORMS OF SMALL TENSORS OVER \mathbb{F}_2

MURRAY R. BREMNER AND JIAXIONG HU

ABSTRACT. We consider multidimensional arrays with at most 27 entries over the field with two elements, and their equivalence classes for the action of the direct product of general linear groups. The possible 3-dimensional formats are $p \times 2 \times 2$ ($p = 2, \dots, 6$), $p \times 3 \times 2$ ($p = 3, 4$), and $3 \times 3 \times 3$; the possible 4-dimensional formats are $p \times 2 \times 2 \times 2$ ($p = 2, 3$). In each case, we compute the orbits for the group action, and then we determine the rank of each orbit. In particular, we determine the maximum rank for these arrays over \mathbb{F}_2 .

1. INTRODUCTION

By a tensor we mean an element of the tensor product of vector spaces. The rank of a tensor is the minimal number of terms in its expression as a sum of simple tensors. For a recent survey, with emphasis on algorithms and applications over fields of characteristic 0, see Kolda and Bader [4]. An equivalent problem is the computational complexity of sets of multilinear forms; see Bürgisser et al. [1].

In the classical case of two vector spaces over any field \mathbb{F} , we can express the problem in terms of $p \times q$ matrices ($p \geq q$). The orbit representatives for the action of $GL_p(\mathbb{F}) \times GL_q(\mathbb{F})$ are the $q + 1$ matrices of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (r = 0, \dots, q),$$

where I_r is the $r \times r$ identity matrix and r is the rank. For more than two vector spaces, the situation is much more complicated.

In this paper, we consider vector spaces over the field \mathbb{F}_2 with two elements; the maximum rank of such a tensor can be higher than expected. For example, Kruskal [5] observed that over \mathbb{R} the rank of a $3 \times 3 \times 3$ tensor is at most 5, but von zur Gathen [7] has shown that over \mathbb{F}_2 there exist $3 \times 3 \times 3$ tensors of rank 6. In this paper we verify that the maximum rank is 6 and exhibit three inequivalent canonical forms of this rank. For tensors of format $p \times q \times 2$, see the recent results by Sumi et al. [6], which complete earlier results of Ja'Ja' [3]. We verify that for tensors of formats $3 \times 3 \times 2$ and $4 \times 3 \times 2$, the maximum rank is 5, and we obtain respectively one and four distinct canonical forms.

Limitations on available computer memory impose a maximum of 27 entries, which restricts our study to the formats $p \times 2 \times 2$ ($p = 2, \dots, 6$), $p \times 3 \times 2$ ($p = 3, 4$), $3 \times 3 \times 3$, and $p \times 2 \times 2 \times 2$ ($p = 2, 3$). In each case, the set of tensors admits an action by the direct product of general linear groups. The canonical form of such a tensor is the lexicographically minimal element of its orbit under this group action.

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format	# tensors	group order	# orbits	max rank
$2 \times 2 \times 2$	256	216	8	3
$3 \times 2 \times 2$	4096	6048	10	3
$4 \times 2 \times 2$	65536	725760	11	4
$5 \times 2 \times 2$	1048576	359976960	11	4
$6 \times 2 \times 2$	16777216	725713551360	11	4
$3 \times 3 \times 2$	262144	169344	21	5
$4 \times 3 \times 2$	16777216	20321280	28	5
$3 \times 3 \times 3$	134217728	28449792	56	6
$2 \times 2 \times 2 \times 2$	65536	31104	31	6
$3 \times 2 \times 2 \times 2$	16777216	217728	213	6

TABLE 1. Summary of results

Since the rank is the same for tensors in the same orbit, once we have found the canonical forms, it is not difficult to find the ranks.

A summary of our results appears in Table 1. In the first seven cases, the group of symmetries is the “small group”: the direct product of general linear groups. In the last three cases, the group of symmetries is the “large group”: the semidirect product of the small group with the permutations of the equal dimensions. Orbits for the action of the small (large) group will be called small (large) orbits.

All computations were done with Maple 16 on a Lenovo ThinkCentre M91p Tower 7052A8U i7-2600 CPU (Quad Core 3.40/3.80GHz) using Windows 7 Professional 64-bit with 16 gigabytes of RAM.

2. PRELIMINARIES

Let V_1, \dots, V_n be finite dimensional vector spaces over a field \mathbb{F} . By a tensor we mean an element of the tensor product $V_1 \otimes \dots \otimes V_n$. A simple tensor is one of the form $v_1 \otimes \dots \otimes v_n$ where $v_i \in V_i$, $v_i \neq 0$ ($i = 1, \dots, n$). Every tensor is a sum of simple tensors, and the rank of a tensor is the minimal number of simple tensors occurring in such a representation. If we choose bases in each V_i then we can identify a tensor X with a multidimensional array $(x_{i_1 \dots i_n})$ of format $d_1 \times \dots \times d_n$ where $d_i = \dim V_i$ ($i = 1, \dots, n$). We also use the term tensor for such an array, in order to avoid confusion with the data structures called arrays in Maple.

The flattening of a tensor $X = (x_{i_1 \dots i_n})$ is the row vector

$$\text{flat}(X) = [x_{1 \dots 1}, \dots, x_{i_1 \dots i_n}, \dots, x_{d_1 \dots d_n}],$$

where the entries are in lex order by subscripts: $i_1 \dots i_n$ precedes $i'_1 \dots i'_n$ if and only if $i_j < i'_j$ where j is the least index for which $i_j \neq i'_j$. Conversely, the unflattening of such a row vector is the corresponding tensor. If \mathbb{F} is the prime field \mathbb{F}_m with m elements, then we can encode a tensor X as the non-negative integer whose representation in base m is $\text{flat}(X)$. Conversely, the decoding of an integer in the range from 0 to $m^{d_1 \dots d_n} - 1$ is the corresponding tensor. The lex order on flattenings coincides with the natural order on integers. The minimal element of a set of tensors is defined in terms of this total order.

The direct product $GL(V_1) \times \dots \times GL(V_n)$ of general linear groups acts on $V_1 \otimes \dots \otimes V_n$ in the natural way. The canonical form of a tensor is the minimal

element in its orbit under this group action. For $\mathbb{F} = \mathbb{F}_m$, the finite group $GL(V)$, $\dim V = d$, has order $(m^d - 1)(m^d - m) \cdots (m^d - m^{d-1})$. If we choose a basis e_1, \dots, e_d for V then elements of $GL(V)$ correspond to $d \times d$ matrices over \mathbb{F}_m . For $m = 2$, the group of invertible $d \times d$ matrices is generated by two elements: the cyclic permutation $e_i \mapsto e_{i+1}$ ($i = 1, \dots, d-1$), $e_d \mapsto e_1$, and the row operation $e_1 \mapsto e_1 + e_2$, $e_i \mapsto e_i$ ($i = 2, \dots, d$). Hence the direct product $GL(V_1) \times \cdots \times GL(V_n)$ is generated by a set \mathcal{G} of $2n$ elements.

For a tensor X over \mathbb{F}_2 , we use the spinning algorithm to compute its orbit. In the following pseudocode, \mathcal{O} is the current value of the orbit, \mathcal{L} contains the new elements computed during the previous iteration, and \mathcal{N} contains the new elements computed during the current iteration:

- (1) $\mathcal{O} \leftarrow \emptyset$; $\mathcal{L} \leftarrow \{X\}$
- (2) while $\mathcal{L} \neq \emptyset$ do:
 - (a) $\mathcal{O} \leftarrow \mathcal{O} \cup \mathcal{L}$
 - (b) $\mathcal{N} \leftarrow \emptyset$; for $Y \in \mathcal{L}$ do for $M \in \mathcal{G}$ do: $\mathcal{N} \leftarrow \mathcal{N} \cup \{M \cdot Y\}$
 - (c) $\mathcal{L} \leftarrow \mathcal{N} \setminus \mathcal{O}$
- (3) return \mathcal{O}

We first create a large Maple array, called **orbitarray**, with $2^N - 1$ entries, where $N = d_1 \cdots d_n$ is the number of entries in the tensors under consideration. The indices of **orbitarray** correspond to nonzero tensors: for an index $i = 1, \dots, 2^N - 1$ we first decode i by writing it as a binary numeral of N bits (adding leading 0s if necessary), and then unflatten this binary numeral to obtain the corresponding tensor. To start, every entry of **orbitarray** is set to 0. We then perform the following iteration:

- (1) $\omega \leftarrow 0$, $i \leftarrow 0$
- (2) while $i < 2^N - 1$ do:
 - (a) $i \leftarrow i + 1$
 - (b) if **orbitarray**[i] = 0 then
 - (i) $\omega \leftarrow \omega + 1$
 - (ii) **findorbit**[i]

Procedure **findorbit** takes the index i , decodes and unflattens it to the corresponding tensor X , uses the spinning algorithm to generate the orbit $\mathcal{O}(X)$, and sets the corresponding entries of **orbitarray** to the orbit index ω . Upon termination, ω equals the total number of orbits for the group action, and **orbitarray** represents the function which assigns to each tensor the index number of its orbit. The natural order of the index numbers of the orbits agrees with the lex order on the minimal elements in the orbits (the canonical forms of the tensors).

The next step is to compute the ranks of the orbits. We create another Maple array, called **linkarray**, of the same size as **orbitarray**. We use the data from **orbitarray** to set entry i of **linkarray** (representing the tensor X) equal to the index j of the next tensor in lex order in the orbit containing X . We then create another Maple array of the same size, called **rankarray**, and initialize every entry to 0. We generate all simple tensors (tensor products of nonzero vectors) and set the corresponding entries of **rankarray** to 1. Each index i for which **rankarray**[i] = 1 represents the encoding of a tensor of rank 1. Let E denote the minimal tensor of rank 1: its flattening is $[0, \dots, 0, 1]$. We then perform the following iteration:

- (1) **oldrank** $\leftarrow 0$, **finished** \leftarrow false

- (2) While not **finished** do:
- (a) **oldrank** \leftarrow **oldrank** + 1, **finished** \leftarrow true
 - (b) For each index i for which **rankarray**[i] = **oldrank**, do:
 - (i) Let X be the unflattening of the decoding of i .
 - (ii) Set $Y \leftarrow X + E$: this amounts to changing the rightmost bit of the flattening of X from 0 to 1 or from 1 to 0.
 - (iii) Let j be the encoding of the flattening of Y . Thus $j = i + 1$ if i is even, and $j = i - 1$ if i is odd.
 - (iv) If **rankarray**[j] = 0, then Y has rank **oldrank** + 1. In this case:
 - Use **linkarray** to store **oldrank** + 1 in every entry of **rankarray** corresponding to the tensors in the orbit of Y .
 - **finished** \leftarrow false

The iteration terminates when every entry of **rankarray** contains a positive integer, which is the rank of the corresponding (nonzero) tensor.

3. FORMATS $p \times 2 \times 2$ ($p = 2, \dots, 6$)

For $p = 2$ there are $2^8 = 256$ tensors, and the group order is $6^3 = 216$. There are 7 nonzero orbits with maximum rank 3. Here is a summary:

rank	0	1	2	3
# orbits	1	1	4	2
# tensors	1	27	162	66
percent	0.3906	10.5469	63.2813	25.7813

For the orbit sizes and canonical forms, see Table 2; the orbits are sorted first by increasing rank, within each rank by increasing orbit size, and within each orbit size by lex order of the canonical forms. This computation took less than 0.1 second. A similar classification appears in [2]; however, that work includes the symmetries obtained by permuting the three directions.

For $p = 3$ there are $2^{12} = 4096$ tensors, and the group order is $168 \cdot 6^2 = 6048$. There are 9 nonzero orbits with maximum rank 3:

rank	0	1	2	3
# orbits	1	1	4	4
# tensors	1	63	1050	2982
percent	0.0244	1.5381	25.6348	72.8027

For the orbit sizes and canonical forms, see Table 3. This computation took less than 1 second.

For $p = 4$ there are $2^{16} = 65536$ tensors, and the group order is $20160 \cdot 6^2 = 725760$. There are 10 nonzero orbits with maximum rank 4:

rank	0	1	2	3	4
# orbits	1	1	4	4	1
# tensors	1	135	5130	40110	20160
percent	0.0015	0.2060	7.8278	61.2030	30.7617

For the orbit sizes and canonical forms, see Table 4. One of the orbits of rank 3 contains almost 35% of the tensors; the single orbit of rank 4 contains almost 31% of the tensors. This computation took less than 10 seconds.

#	rank	size	canonical form									
1	1	27	1
2	2	18	1	1	.	.
3	2	18	1	.	.	1	.	.
4	2	18	1	.	1	.	.	.
5	2	108	1	1
6	3	12	.	1	1	.	1	.	1	1	.	.
7	3	54	1	.	1	1	.	.

TABLE 2. Small orbits of $2 \times 2 \times 2$ tensors

#	rank	size	canonical form									
1	1	63	1
2	2	42	1	1
3	2	126	1	.	.	1
4	2	126	1	.	1	.
5	2	756	1	1	.	.
6	3	84	1	1	.	1	1
7	3	378	1	.	1	1
8	3	1008	.	.	.	1	.	1	1	.	1	.
9	3	1512	.	.	.	1	.	.	1	.	1	.

TABLE 3. Small orbits of $3 \times 2 \times 2$ tensors

#	rank	size	canonical form									
1	1	135	1
2	2	90	1
3	2	630	1	.
4	2	630	1	.
5	2	3780	1	.
6	3	420	1	1	.
7	3	1890	1	.
8	3	15120	1	.	1	.
9	3	22680	1	.	1	.
10	4	20160	.	.	.	1	.	.	1	.	1	.

TABLE 4. Small orbits of $4 \times 2 \times 2$ tensors

For $p = 5$ and $p = 6$ we obtain the same canonical forms with the addition of more leading 0s in the flattenings of the tensors; thus in both cases there are 10 nonzero orbits with maximum rank 4.

For $p = 5$ there are $2^{20} = 1048576$ tensors; the group order is $9999360 \cdot 6^2 = 359976960$. Here is a summary:

rank	0	1	2	3	4
# orbits	1	1	4	4	1
# tensors	1	279	22506	400830	624960
percent	0.0001	0.0266	2.1463	38.2261	59.6008

The orbit sizes, using the order of Table 4 (with four more leading 0s), are:

279, 186, 2790, 2790, 16740, 1860, 8370, 156240, 234360, 624960.

The single orbit of rank 4 contains almost 60% of the tensors. This computation took less than 4 minutes.

For $p = 6$ there are $2^{24} = 16777216$ tensors; the group order is $20158709760 \cdot 6^2 = 725713551360$. Here is a summary:

rank	0	1	2	3	4
# orbits	1	1	4	4	1
# tensors	1	567	94122	3558366	13124160
percent	0.0000	0.0034	0.5610	21.2095	78.2261

The orbit sizes, using the order of Table 4 (with eight more leading 0s), are:

567, 378, 11718, 11718, 70308, 7812, 35154, 1406160, 2109240, 13124160.

The single orbit of rank 4 contains more than 78% of the tensors. This computation took just over 75 minutes.

These results suggest the following conjectures for $p \times 2 \times 2$ tensors with $p \geq 4$.

Conjecture 1. *For $p \geq 4$, there are 10 canonical forms for nonzero $p \times 2 \times 2$ tensors over \mathbb{F}_2 . For each orbit, the rank and the horizontal 2×2 slices of the minimal element (obtained by fixing the first subscript) are given in Table 5. The leftmost empty slice indicates the appropriate number of zero 2×2 slices.*

Conjecture 2. *As p becomes arbitrarily large, the set of $p \times 2 \times 2$ tensors over \mathbb{F}_2 contains a single orbit \mathcal{O} of rank 4 which contains almost all of the 2^{4p} tensors:*

$$\lim_{p \rightarrow \infty} \frac{|\mathcal{O}|}{2^{4p}} = 1.$$

4. FORMATS $p \times 3 \times 2$ ($p = 3, 4$)

For $p = 3$ there are $2^{18} = 262144$ tensors, and the group order is $168^2 \cdot 6 = 169344$. There are 20 nonzero orbits with maximum rank 5:

rank	0	1	2	3	4	5
# orbits	1	1	4	9	5	1
# tensors	1	147	6762	95466	151704	8064
percent	0.0004	0.0561	2.5795	36.4174	57.8705	3.0762

For the orbit sizes and canonical forms, see Table 6. This computation took less than 30 seconds.

#	rank	canonical form
1	1	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array} \right]$
2	2	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{array} \right]$
3	2	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{array} \right]$
4	2	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array} \right]$
5	2	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array} \right]$
6	3	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \end{array} \right]$
7	3	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{array} \right]$
8	3	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{array} \right]$
9	3	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{array} \right]$
10	4	$\left[\begin{array}{c cc cc cc} & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{array} \right]$

TABLE 5. Canonical forms for Conjecture 1

For $p = 4$ there are $2^{24} = 16777216$ tensors, and the group order is $20160 \cdot 168 \cdot 6 = 20321280$. There are 27 nonzero orbits with maximum rank 5:

rank	0	1	2	3	4	5
# orbits	1	1	4	9	9	4
# tensors	1	315	32970	1223250	10460520	5060160
percent	0.0000	0.0019	0.1965	7.2911	62.3496	30.1609

For the orbit sizes and canonical forms, see Table 7. This computation took less than 45 minutes.

5. FORMAT $3 \times 3 \times 3$

In this and the following section, the formats have three (or four) equal dimensions. We exploit this symmetry in order to reduce the number of orbits to a manageable size. For $3 \times 3 \times 3$ tensors, the large group acting on the tensors is

$$G = (GL(3, \mathbb{F}_2) \times GL(3, \mathbb{F}_2) \times GL(3, \mathbb{F}_2)) \rtimes S_3,$$

where the symmetric group S_3 permutes the three directions. As in the previous sections, we first compute the small orbits obtained by the action of the direct product of general linear groups. We then apply the permutations to determine which small orbits combine to make a single large orbit under the action of G . Given the canonical form X of a small orbit \mathcal{O} with index number i , we apply the

#	rank	size	canonical form															
1	1	147	1
2	2	294	1	1
3	2	294	1	.	.	.	1	.
4	2	882	1	.	.	.	1	.
5	2	5292	1	.	.	1	.	.
6	3	504	1	.	.	.	1	.	.	.	1	.	.
7	3	588	1	1	.	.	.	1	1
8	3	2646	1	.	.	.	1	1
9	3	7056	1	1	.	.	1	1	.
10	3	7056	1	.	.	.	1	1	.	.	.	1	.
11	3	10584	1	.	1	1	.	.
12	3	10584	1	1	.	.	.	1	.
13	3	28224	1	.	.	.	1	.	.	1	.	.	.
14	3	28224	1	.	.	1	.	.	.	1	1	.	.
15	4	7056	1	1	.	.	1	1	.
16	4	10584	1	.	.	.	1	.	.	.	1	.	1
17	4	21168	1	.	.	.	1	1	.	.	1	1	.
18	4	28224	1	.	1	1	.	.	.	1	.	1	1
19	4	84672	1	.	.	.	1	1	.	1	.	.	.
20	5	8064	.	.	.	1	1	.	.	1	1	.	.	.	1	.	.	1

TABLE 6. Small orbits of $3 \times 3 \times 2$ tensors

elements of S_3 to obtain tensors $X_1 = X, \dots, X_6$. We then use the Maple arrays, which we have already computed, to find the index numbers $i_1 = i, \dots, i_6$ of the small orbits containing these tensors. We conclude that the union $\mathcal{O}_{i_1} \cup \dots \cup \mathcal{O}_{i_6}$ is a large orbit for the action of G . The canonical form of the tensors in this large orbit is the smallest (in lex order) of the canonical forms of $\mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_6}$.

With format $3 \times 3 \times 3$ there are $2^{27} = 134217728$ tensors. The small group order is $168^3 = 4741632$, and the large group order is $6 \cdot 168^3 = 28449792$. There are 115 (nonzero) small orbits and 55 (nonzero) large orbits, with maximum rank 6:

rank	0	1	2	3	4	5	6
# small	1	1	4	18	44	45	3
# large	1	1	2	8	18	23	3
# tensors	1	343	43218	2372286	47506872	83670048	624960
percent	0.0000	0.0003	0.0322	1.7675	35.3954	62.3390	0.4656

For the large orbit sizes and canonical forms, see Table 8. This computation took just under 282 minutes.

6. FORMATS $p \times 2 \times 2 \times 2$ ($p = 2, 3$)

For $p = 2$ the classification appears in [2]; there are $2^{16} = 65536$ tensors. The small group order is $6^4 = 1296$, and the large group order is $24 \cdot 6^4 = 31104$ (we include all permutations of the four equal directions).

#	rank	size	canonical form									
1	1	315	1
2	2	630	1	1
3	2	1470	1	.	1
4	2	4410	1	.	1
5	2	26460	1	.	1
6	3	2940	1	1	.	1
7	3	7560	1	.	1	.	1
8	3	13230	1	.	1	1
9	3	35280	1	1	.	1
10	3	52920	1	.	1	1
11	3	105840	1	.	1	1	.
12	3	158760	1	.	.	1	.
13	3	423360	1	.	1	.	1
14	3	423360	1	.	1	.	1
15	4	105840	1	.	1	.	1
16	4	141120	.	.	.	1	.	1	.	1	.	1
17	4	158760	1	.	1	.	1
18	4	317520	1	1	1	.	1
19	4	423360	1	1	1	.	1
20	4	423360	.	.	.	1	.	1	.	1	.	1
21	4	1270080	1	1	1	.	1
22	4	2540160	.	.	.	1	.	1	.	1	.	1
23	4	5080320	.	.	.	1	.	1	.	1	.	1
24	5	120960	1	1	.	1	1	.
25	5	282240	.	.	.	1	.	1	.	1	1	.
26	5	1270080	.	.	.	1	.	1	.	1	1	.
27	5	3386880	.	.	.	1	.	1	1	.	1	.

TABLE 7. Small orbits of $4 \times 3 \times 2$ tensors

For $p = 3$, the large group acting on the tensors is

$$G = (GL(3, \mathbb{F}_2) \times GL(2, \mathbb{F}_2) \times GL(2, \mathbb{F}_2) \times GL(2, \mathbb{F}_2)) \rtimes S_3,$$

where S_3 permutes the last three directions. We follow the same approach as the previous section, but now there are $2^{24} = 16777216$ tensors. The small group order is $168 \cdot 6^3 = 36288$, and the large group order is $6 \cdot 168 \cdot 6^3 = 217728$. There are 696 (nonzero) small orbits and 212 (nonzero) large orbits, with maximum rank 6:

rank	0	1	2	3	4	5	6
# small	1	1	11	49	234	367	34
# large	1	1	5	21	72	100	13
# tensors	1	189	13608	434028	5143446	10460016	725928
percent	0.0000	0.0011	0.0811	2.5870	30.6573	62.3466	4.3269

For the large orbit sizes and canonical forms, see Tables 9 and 10. This computation took less than 40 minutes.

#	rank	size	canonical form
1	1	343 1
2	2	6174 1 . 1 .
3	2	37044 1 1
4	3	3528 1 . 1 . 1 . .
5	3	4116 1 . 1 . . . 1 . . 1 1
6	3	18522 1 1 . 1 .
7	3	148176 1 . 1 . . . 1 . 1 . .
8	3	222264 1 1 . 1 . .
9	3	592704 1 . 1 . 1
10	3	592704 1 . 1 . 1 1 .
11	3	790272 1 1 1
12	4	148176 1 . 1 . . . 1 . . . 1 . . .
13	4	197568 1 . 1 . . . 1 . . . 1 . 1 1 . . 1 .
14	4	222264 1 1 . . . 1 . 1 . 1 . 1 .
15	4	263424 1 . 1 . 1 1 . . 1 1
16	4	444528 1 . 1 . . . 1 . 1 . 1 . . 1 . .
17	4	592704 1 . 1 . 1 . . . 1 . . . 1 1
18	4	1185408 1 1 . 1 . 1
19	4	1778112 1 . 1 . . . 1 . 1 . . 1 1
20	4	1778112 1 1 . . . 1 1
21	4	1778112 1 1 1 1 1 . . 1 . .
22	4	2370816 1 . 1 . 1 1 . . 1 1 . . 1 .
23	4	2370816 1 . 1 . 1 1 . . 1 1 1 . 1 .
24	4	3556224 1 1 . 1 . 1 1 .
25	4	4741632 1 1 1 1 .
26	4	4741632 1 1 . 1 . . 1 1 . . .
27	4	7112448 1 1 . 1 . 1 1
28	4	7112448 1 1 1 1 . 1 .
29	4	7112448 1 1 . 1 . . . 1 1 1
30	5	28224 1 . 1 . . . 1 . . . 1 . . . 1 . 1
31	5	148176 1 1 1 . . . 1 . 1 . 1 . .
32	5	148176 1 1 1 . 1 . . . 1 . 1 . .
33	5	169344 1 1 . 1 . 1 . . . 1 . 1 . . . 1 1
34	5	592704 1 . 1 . . . 1 . . . 1 . . . 1 . 1 1 . . 1 1
35	5	1185408 1 . 1 . . . 1 . . . 1 . . . 1 . 1 . . . 1 .
36	5	1580544 1 . 1 . . . 1 . . . 1 . . . 1 . . 1 1
37	5	1580544 1 . 1 . . . 1 . . . 1 . . . 1 . . 1 1 . . . 1
38	5	1778112 1 1 1 . 1 . . . 1 1
39	5	1778112 1 1 1 . 1 . . . 1 1 . . 1 1 .
40	5	2370816 1 . 1 . . . 1 . . . 1 . . . 1 . . 1 1 . 1 . .
41	5	2370816 1 . 1 . . . 1 . 1 . 1 . . . 1 1 .
42	5	2370816 1 . 1 . . . 1 . 1 . 1 . . . 1 . . . 1 . 1 .
43	5	3556224 1 1 1 . 1 . . 1
44	5	4741632 1 . 1 . 1 . 1 1 1 1 1 1 . .
45	5	4741632 1 . 1 . . . 1 . 1 . 1 . . . 1 . 1 . . 1 . .
46	5	4741632 1 . 1 . . . 1 . 1 . 1 . . . 1 1 .
47	5	4741632 1 . 1 . . . 1 . 1 . 1 . . . 1 1 1
48	5	4741632 1 . 1 . . . 1 . 1 . 1 . . . 1 1 .
49	5	4741632 1 . 1 . . . 1 . 1 . 1 . . . 1 . . . 1 1 .
50	5	7112448 1 . . . 1 . 1 . 1 . . . 1 1 . 1 .
51	5	14224896 1 1 . 1 . 1 . . . 1 1 . 1 .
52	5	14224896 1 . . . 1 . 1 . 1 . . . 1 . 1
53	6	32256	. . 1 . 1 . 1 . . . 1 . 1 . . . 1 1 1 . . . 1 1 1 1 .
54	6	197568 1 . 1 . . . 1 . . . 1 . . . 1 . 1 . . . 1 .
55	6	395136 1 . 1 . . . 1 . 1 . 1 . . . 1 . 1 . . . 1 1

TABLE 8. Large orbits of $3 \times 3 \times 3$ tensors

#	rank	size	canonical form															
1	1	189	1
2	2	378	1	1
3	2	756	1	1	.
4	2	1134	1	1	.
5	2	4536	1	1	.
6	2	6804	1	1	.
7	3	84	1	1	1
8	3	378	1	1	1
9	3	756	1	1	1
10	3	1512	1	1	1
11	3	3402	1	1	1
12	3	4536	1	1	1
13	3	4536	1	1	1
14	3	6048	1	1	1
15	3	9072	1	1	1
16	3	13608	1	1	1
17	3	13608	1	1	1
18	3	13608	1	1	1
19	3	13608	1	1	1
20	3	13608	1	1	1
21	3	27216	1	1	1
22	3	27216	1	1	1
23	3	27216	1	1	1
24	3	36288	1	1	1
25	3	54432	1	1	1
26	3	54432	1	1	1
27	3	108864	1	1	1
28	4	756	1	1	1
29	4	1134	1	1	1
30	4	4536	1	1	1
31	4	4536	1	1	1
32	4	4536	1	1	1
33	4	4536	1	1	1
34	4	6804	1	1	1
35	4	6804	1	1	1
36	4	6804	1	1	1
37	4	9072	1	1	1
38	4	9072	1	1	1
39	4	13608	1	1	1
40	4	13608	1	1	1
41	4	13608	1	1	1
42	4	13608	1	1	1
43	4	13608	1	1	1
44	4	13608	1	1	1
45	4	13608	1	1	1
46	4	27216	1	1	1
47	4	27216	1	1	1
48	4	27216	1	1	1
49	4	27216	1	1	1
50	4	27216	1	1	1
51	4	27216	1	1	1
52	4	27216	1	1	1
53	4	27216	1	1	1
54	4	36288	1	1	1
55	4	36288	1	1	1
56	4	36288	1	1	1
57	4	36288	1	1	1
58	4	54432	1	1	1
59	4	54432	1	1	1
60	4	54432	1	1	1
61	4	54432	1	1	1
62	4	54432	1	1	1
63	4	54432	1	1	1
64	4	54432	1	1	1
65	4	54432	1	1	1
66	4	54432	1	1	1
67	4	54432	1	1	1
68	4	54432	1	1	1
69	4	54432	1	1	1
70	4	54432	1	1	1
71	4	108864	1	1	1
72	4	108864	1	1	1
73	4	108864	1	1	1
74	4	108864	1	1	1
75	4	108864	1	1	1
76	4	108864	1	1	1
77	4	108864	1	1	1
78	4	108864	1	1	1
79	4	108864	1	1	1
80	4	108864	1	1	1
81	4	108864	1	1	1
82	4	108864	1	1	1
83	4	108864	1	1	1
84	4	108864	1	1	1
85	4	108864	1	1	1
86	4	108864	1	1	1
87	4	108864	1	1	1
88	4	108864	1	1	1
89	4	108864	1	1	1
90	4	108864	1	1	1
91	4	108864	1	1	1
92	4	108864	1	1	1
93	4	217728	1	1	1
94	4	217728	1	1	1
95	4	217728	1	1	1
96	4	217728	1	1	1
97	4	217728	1	1	1
98	4	217728	1	1	1
99	4	217728	1	1	1
100	5	2268	1	1	1
101	5	2268	1	1	1
102	5	4536	1	1	1
103	5	9072	1	1	1
104	5	9072	1	1	1
105	5	9072	1	1	1

TABLE 9. Large orbits of $3 \times 2 \times 2 \times 2$ tensors, part 1

TABLE 10. Large orbits of $3 \times 2 \times 2 \times 2$ tensors, part 2

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SASKATCHEWAN, CANADA
E-mail address: bremner@math.usask.ca

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, CANADA
E-mail address: hujiaxiong@gmail.com